Parameter rigid actions of simply connected nilpotent Lie groups

Hirokazu Maruhashi *
Department of Mathematics, Kyoto University

Abstract

We show that for a locally free C^∞-action of a connected and simply connected nilpotent Lie group on a compact manifold, if every real valued cocycle is cohomologous to a constant cocycle, then the action is parameter rigid. The converse is true if the action has a dense orbit. Using this, we construct parameter rigid actions of simply connected nilpotent Lie groups whose Lie algebras admit rational structures with graduations. This generalizes the results of dos Santos [8] concerning the Heisenberg groups.

1 Introduction

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and M a C^∞-manifold without boundary. Let $\rho : M \times G \to M$ be a C^∞ right action. We call ρ *locally free* if every isotropy subgroup of ρ is discrete in G. Assume that ρ is locally free. Then we have the orbit foliation $\mathcal{F}$ of ρ whose tangent bundle $T\mathcal{F}$ is naturally isomorphic to a trivial bundle $M \times \mathfrak{g}$.

The action ρ is *parameter rigid* if any action ρ' of G on M with the same orbit foliation $\mathcal{F}$ is C^∞-conjugate to ρ, more precisely, there exist an automorphism Φ of G and a C^∞-diffeomorphism F of M which preserves each leaf of $\mathcal{F}$ and homotopic to identity through C^∞-maps preserving each leaf of $\mathcal{F}$ such that

$$F(\rho(x, g)) = \rho'(F(x), \Phi(g))$$

for all $x \in M$ and $g \in G$.

Parameter rigidity of actions has been studied by several authors, for instance, Katok and Spatzier [3], Matsumoto and Mitsumatsu [4], Mieczkowski [5], dos Santos [8] and Ramírez [7]. Most of known examples of parameter rigid actions are those of abelian groups and nonabelian actions have not been considered so much.

*Research Fellow of the Japan Society for the Promotion of Science

Parameter rigidity is closely related to cocycles over actions. Now suppose G is contractible and M is compact. Let H be a Lie group. A C^∞-map $c : M \times G \to H$ is called a *H-valued cocycle over ρ* if c satisfies

$$c(x, gg') = c(x, g)c(\rho(x, g), g')$$

for all $x \in M$ and $g, g' \in G$.

A cocycle c is *constant* if $c(x, g)$ is independent of x. A constant cocycle is just a homomorphism $G \to H$.

H-valued cocycles c, c' are *cohomologous* if there exists a C^∞-map $P : M \to H$ such that

$$c(x, g) = P(x)^{-1}c'(x, g)P(\rho(x, g))$$

for all $x \in M$ and $g \in G$.

The action ρ is *H-valued cocycle rigid* if every H-valued cocycle over ρ is cohomologous to a constant cocycle.

Proposition 1 ([4]). *If ρ is G-valued cocycle rigid, then it is parameter rigid.*

Remark. In [4] Matsumoto and Mitsumatsu assume that ρ has at least one trivial isotropy subgroup, but this assumption is not necessary.

Proposition 2 ([4]). *When $G = \mathbb{R}^n$, the following are equivalent:*

1. *ρ is $\mathbb{R}$-valued cocycle rigid.*
2. *ρ is $\mathbb{R}^n$-valued cocycle rigid.*
3. *ρ is parameter rigid.*

Remark. The equivalence of the first two conditions is obvious.

In this paper we consider actions of simply connected nilpotent Lie groups. In [8], dos Santos proved that for actions of a Heisenberg group H_n, $\mathbb{R}$-valued cocycle rigidity implies H_n-valued cocycle rigidity and using this, he constructed parameter rigid actions of Heisenberg groups. To the best of my knowledge these are the only known nontrivial parameter rigid actions of nonabelian nilpotent Lie groups. We prove the following.

Theorem 1. *Let N be a connected and simply connected nilpotent Lie group, M a compact manifold and ρ a locally free C^∞-action of N on M. Then, the following are equivalent:*

1. *ρ is $\mathbb{R}$-valued cocycle rigid.*
2. *ρ is N-valued cocycle rigid.*
3. *ρ is parameter rigid and every orbitwise constant real valued C^∞-function of ρ on M is constant on M.*

This theorem enables us to construct parameter rigid actions of nilpotent Lie groups. The most interesting one is the following.

Theorem 2 ([7]). *Let N denote the group of all upper triangular real matrices with 1 on the diagonal, Γ a cocompact lattice of $\mathrm{SL}(n, \mathbb{R})$ and ρ the action of N on $\Gamma \backslash \mathrm{SL}(n, \mathbb{R})$ by right multiplication. If $n \geq 4$, ρ is $\mathbb{R}$-valued cocycle rigid.*

Remark. In [7], Ramírez proved more general theorems.

Corollary. *The above action ρ is parameter rigid.*

In Section 4 we construct parameter rigid actions of nilpotent groups using Theorem 1. It is a generalization of dos Santos' example. Let N be a connected and simply connected nilpotent Lie group and Γ, Λ be lattices in N. Consider the action of Λ on $\Gamma \backslash N$ by right multiplication and let $\tilde{\rho}$ be its suspended action of N.

Theorem 3. *If Λ is Diophantine with respect to Γ, then the action $\tilde{\rho}$ of N is parameter rigid.*

For the definition of Diophantine lattices, see Section 4.

2 Preliminaries

Let G be a contractible Lie group with Lie algebra $\mathfrak{g}$, M a compact manifold and ρ a locally free action of G on M with orbit foliation $\mathcal{F}$. Let H be a Lie group with Lie algebra $\mathfrak{h}$. We denote by $\Omega^p(\mathcal{F}, \mathfrak{h})$ the set of all C^∞-sections of $\mathrm{Hom}(\bigwedge^p T\mathcal{F}, \mathfrak{h})$. The exterior derivative

$$d_{\mathcal{F}} : \Omega^p(\mathcal{F}, \mathfrak{h}) \to \Omega^{p+1}(\mathcal{F}, \mathfrak{h})$$

is defined since $T\mathcal{F}$ is integrable.

By differentiating, H-valued cocycles over ρ are in one-to-one correspondence with $\mathfrak{h}$-valued leafwise one forms $\omega \in \Omega^1(\mathcal{F}, \mathfrak{h})$ such that

$$d_{\mathcal{F}}\omega + [\omega, \omega] = 0.$$

Proposition 3. *Let c_1, c_2 be H-valued cocycles over ρ and let ω_1, ω_2 be corresponding differential forms. For a C^∞-map $P : M \to H$, the following are equivalent:*

1. *$c_1(x, g) = P(x)^{-1} c_2(x, g) P(\rho(x, g))$ for all $x \in M$ and $g \in G$.*
2. *$\omega_1 = \mathrm{Ad}(P^{-1})\omega_2 + P^*\theta$ where $\theta \in \Omega^1(H, \mathfrak{h})$ is the left Maurer-Cartan form on H.*

Corollary ([4]). *The following are equivalent:*

1. *ρ is G-valued cocycle rigid.*
2. *For each $\omega \in \Omega^1(\mathcal{F}, \mathfrak{g})$ such that $d_{\mathcal{F}}\omega + [\omega, \omega] = 0$, there exist a endomorphism $\Phi : \mathfrak{g} \to \mathfrak{g}$ of Lie algebra and a C^∞-map $P : M \to G$ such that*
$$\omega = \mathrm{Ad}(P^{-1})\Phi + P^*\theta.$$

Proposition 3 is obtained by examining the proof of Corollary 2 in [4]. In this paper, we will identify a cocycle with its corresponding differential form.

Let us consider real valued cocycles. A real valued cocycle over ρ is given by $\omega \in \Omega^1(\mathcal{F}, \mathbb{R})$ satisfying $d_{\mathcal{F}}\omega = 0$. Two real valued cocycles ω_1, ω_2 are cohomologous if and only if $\omega_1 = \omega_2 + d_{\mathcal{F}}P$ for some C^∞-function $P : M \to \mathbb{R}$. Leafwise cohomology $H^*(\mathcal{F})$ of $\mathcal{F}$ is the cohomology of the cochain complex $(\Omega^*(\mathcal{F}, \mathbb{R}), d_{\mathcal{F}})$. Thus $H^1(\mathcal{F})$ is the set of all equivalence classes of real valued cocycles.

The identification $T\mathcal{F} \simeq M \times \mathfrak{g}$ induces a map $H^*(\mathfrak{g}) \to H^*(\mathcal{F})$ where $H^*(\mathfrak{g})$ is the cohomology of the Lie algebra $\mathfrak{g}$. By the compactness of M, this map is injective on $H^1(\mathfrak{g})$. Hence we identify $H^1(\mathfrak{g})$ with its image. Note that $H^1(\mathfrak{g})$ is the set of all equivalence classes of constant real valued cocycles. Thus real valued cocycle rigidity is equivalent to $H^1(\mathcal{F}) = H^1(\mathfrak{g})$.

Notice that $H^0(\mathcal{F})$ is the set of leafwise constant real valued C^∞-functions of $\mathcal{F}$ on M and $H^0(\mathfrak{g})$ consists of constant functions on M. Therefore the equivalence of 1 and 3 in Theorem 1 can be stated as follows: $H^1(\mathcal{F}) = H^1(\mathfrak{n})$ if and only if ρ is parameter rigid and $H^0(\mathcal{F}) = H^0(\mathfrak{n})$.

3 Proof of Theorem 1

Let N be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$, M a compact manifold and ρ a locally free action of N on M with orbit foliation $\mathcal{F}$.

We first prove that N-valued cocycle rigidity implies real valued cocycle rigidity. There exist closed subgroups N' and A of N such that $N' \triangleleft N$, $N = N' \rtimes A$ and $A \simeq \mathbb{R}$. Let c be any real valued cocycle over ρ. We regard c as a N-valued cocycle over ρ via the inclusion $\mathbb{R} \simeq A \hookrightarrow N$. By N-valued cocycle rigidity, there exist an endomorphism Φ of N and a C^∞-map $P : M \to N$ such that $c(x, g) = P(x)^{-1}\Phi(g)P(\rho(x, g))$ for all $x \in M$ and $g \in N$. Applying the natural projection $\pi : N \to A \simeq \mathbb{R}$, we obtain $c(x, g) = (\pi \circ P)(x)^{-1}(\pi \circ \Phi)(g)(\pi \circ P)(\rho(x, g))$. Thus c is cohomologous to a constant cocycle $\pi \circ \Phi$.

Next we assume $H^1(\mathcal{F}) = H^1(\mathfrak{n})$ and prove N-valued cocycle rigidity. We need the following two lemmata.

Lemma 1. *Let V be a finite dimensional real vector space. Assume that $\omega \in \Omega^1(\mathcal{F}, V)$ satisfies the equation $d_{\mathcal{F}}\omega = \varphi$, where $\varphi \in \mathrm{Hom}(\bigwedge^2 \mathfrak{n}, V)$ is a constant leafwise two form. Then there exists a constant leafwise one form $\psi \in \mathrm{Hom}(\mathfrak{n}, V)$ with $\varphi = d_{\mathcal{F}}\psi$.*

Proof. Since N is amenable, there exists a N-invariant Borel probability measure μ on M. Define $\psi \in \mathrm{Hom}(\mathfrak{n}, V)$ by

$$\psi(X) = \int_M \omega(X) d\mu$$

where $X \in \mathfrak{n}$. Since $\varphi(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$ for all $X, Y \in \mathfrak{n}$, we obtain

$$\varphi(X, Y) = -\int_M \omega([X, Y]) d\mu.$$

Thus

$$d_{\mathcal{F}}\psi(X,Y) = -\psi([X,Y]) = -\int_M \omega([X,Y])d\mu = \varphi(X,Y),$$

hence $d_{\mathcal{F}}\psi = \varphi$. $\square$

Set $\mathfrak{n}^1 = \mathfrak{n}$, $\mathfrak{n}^i = [\mathfrak{n}, \mathfrak{n}^{i-1}]$. Then $\mathfrak{n}^s \neq 0$, $\mathfrak{n}^{s+1} = 0$ for some s. For each $1 \leq i \leq s$, choose a subspace V_i with $\mathfrak{n}^i = V_i \oplus \mathfrak{n}^{i+1}$, so that $\mathfrak{n} = \bigoplus_{i=1}^s V_i$.

Lemma 2. *Let $\omega \in \Omega^1(\mathcal{F}, \mathfrak{n})$ be such that $d_{\mathcal{F}}\omega + [\omega, \omega] = 0$. Decompose ω as*

$$\omega = \xi + \omega_k + \omega_{k+1}$$

where $\xi \in \Omega^1(\mathcal{F}, \bigoplus_{i=1}^{k-1} V_i)$, $\omega_k \in \Omega^1(\mathcal{F}, V_k)$ and $\omega_{k+1} \in \Omega^1(\mathcal{F}, \mathfrak{n}^{k+1})$. If ξ is constant, then there exists $\omega' \in \Omega^1(\mathcal{F}, \mathfrak{n})$ with $d_{\mathcal{F}}\omega' + [\omega', \omega'] = 0$ which is cohomologous to ω and such that

$$\omega' = \xi' + \omega'_{k+1}$$

where $\xi' \in \Omega^1(\mathcal{F}, \bigoplus_{i=1}^k V_i)$ is constant and $\omega'_{k+1} \in \Omega(\mathcal{F}, \mathfrak{n}^{k+1})$.

Proof. By cocycle equation,

$$0 = d_{\mathcal{F}}\xi + d_{\mathcal{F}}\omega_k + d_{\mathcal{F}}\omega_{k+1} + [\xi, \xi] + \text{an element of } \Omega^2(\mathcal{F}, \mathfrak{n}^{k+1}).$$

Comparing V_k component, we see that $d_{\mathcal{F}}\omega_k$ is constant. Hence by Lemma 1, $d_{\mathcal{F}}\omega_k = d_{\mathcal{F}}\psi$ for some $\psi \in \mathrm{Hom}(\mathfrak{n}, V_k)$. Since we are assuming that $H^1(\mathcal{F}) = H^1(\mathfrak{n})$, there exists $\psi' \in \mathrm{Hom}(\mathfrak{n}, V_k)$ and C^∞-map $h : M \to V_k$ such that

$$\omega_k = \psi + \psi' + d_{\mathcal{F}}h.$$

Put $P = e^h : M \to N$. Let $x \in M$ and $X \in T_x\mathcal{F}$. Choose a path $x(t)$ such that $X = \frac{d}{dt}x(t)\big|_{t=0}$. Let $\theta \in \Omega^1(N, \mathfrak{n})$ be the left Maurer-Cartan form on N. Then

$$\begin{aligned}
P^*\theta(X) &= \frac{d}{dt}P(x)^{-1}P(x(t))\Big|_{t=0} = \frac{d}{dt}e^{-h(x)}e^{h(x(t))}\Big|_{t=0} \\
&= \frac{d}{dt}\exp(-h(x) + h(x(t)) + \text{an element of } \mathfrak{n}^{k+1})\Big|_{t=0} \\
&= d_{\mathcal{F}}h(X) + \text{an element of } \mathfrak{n}^{k+1}.
\end{aligned}$$

Thus $P^*\theta = d_{\mathcal{F}}h +$an element of $\Omega^1(\mathcal{F}, \mathfrak{n}^{k+1})$. Note that $\mathrm{Ad}(P^{-1}) = \exp \mathrm{ad}(-h)$ is identity on $\bigoplus_{i=1}^k V_i$ and preserves $\mathfrak{n}^{k+1}$. Hence

$$\begin{aligned}
\omega - P^*\theta &= \xi + \psi + \psi' + \text{an element of } \Omega^1(\mathcal{F}, \mathfrak{n}^{k+1}) \\
&= \mathrm{Ad}(P^{-1})(\xi + \psi + \psi' + \text{an element of } \Omega^1(\mathcal{F}, \mathfrak{n}^{k+1})).
\end{aligned}$$

$\square$

Let ω be any N-valued cocycle. Using Lemma 2, we can exchange ω for a cohomologous cocycle whose V_1-component is constant. Applying Lemma 2 repeatedly, we eventually get a constant cocycle cohomologous to ω. This proves N-valued cocycle rigidity.

Next we assume that ρ is parameter rigid and $H^0(\mathcal{F}) = H^0(\mathfrak{n})$. Let $\mathfrak{n}^i$ and V_i be as above. Note that $\mathfrak{n}^s$ is central in $\mathfrak{n}$. Fix a nonzero $Z \in \mathfrak{n}^s$.

Let $[\omega] \in H^1(\mathcal{F})$. Let ω_0 be the N-valued cocycle over ρ corresponding to the constant cocycle $\mathrm{id} : N \to N$. We call ω_0 *the canonical 1-form of* ρ. Fix a $\epsilon > 0$ and put $\eta := \omega_0 + \epsilon\omega Z$. η is an N-valued cocycle over ρ since

$$d_{\mathcal{F}}\eta + [\eta, \eta] = d_{\mathcal{F}}\omega_0 + \epsilon(d_{\mathcal{F}}\omega)Z + [\omega_0, \omega_0] = 0.$$

Since M is compact, we can assume $\eta_x : T_x\mathcal{F} \to \mathfrak{n}$ is bijective for all $x \in M$ by choosing $\epsilon > 0$ small. There exists a unique action ρ' of N on M whose orbit foliation is $\mathcal{F}$ and canonical 1-form is η. See [1]. By parameter rigidity ρ' is conjugate to ρ. Thus there exist a C^∞-map $P : M \to N$ and an automorphism Φ of N satisfying

$$\omega_0 + \epsilon\omega Z = \mathrm{Ad}(P^{-1})\Phi_*\omega_0 + P^*\theta. \tag{1}$$

Note that $\log : N \to \mathfrak{n}$ is defined since N is simply connected and nilpotent. Let us decompose $\omega_0 = \sum_{i=1}^s \omega_{0i}$, $\Phi_*\omega_0 = \sum_{i=1}^s \omega'_{0i}$ and $\log P = \sum_{i=1}^s P_i$ according to the decomposition $\mathfrak{n} = \bigoplus_{i=1}^s V_i$.

Lemma 3. *Assume that* $P_1 = \cdots = P_{k-1} = 0$ *i.e.* $\log P \in \mathfrak{n}^k$.

1. *If* $k < s$, *then there exist a* C^∞*-map* $Q : M \to N$ *and an automorphism* Ψ *of* N *such that*

$$\omega_0 + \epsilon\omega Z = \mathrm{Ad}(Q^{-1})\Psi_*\omega_0 + Q^*\theta$$

   and $Q_1 = \cdots = Q_k = 0$ *where* $\log Q = \sum_{i=1}^s Q_i$.

2. *If* $k = s$, *then* ω *is cohomologous to a constant cocycle.*

Proof. For all $X = \frac{d}{dt}x(t)\big|_{t=0} \in T_x\mathcal{F}$,

$$\begin{aligned}
P^*\theta(X) &= \frac{d}{dt}P(x)^{-1}P(x(t))\Big|_{t=0} = \frac{d}{dt}\exp\left(-\sum_{i=k}^s P_i(x)\right)\exp\left(\sum_{i=k}^s P_i(x(t))\right)\Big|_{t=0} \\
&= \frac{d}{dt}\exp\left\{\sum_{i=k}^s (P_i(x(t)) - P_i(x)) + \text{an element of } \mathfrak{n}^{k+1}\right\}\Big|_{t=0} \\
&= \frac{d}{dt}\exp\left(P_k(x(t)) - P_k(x) + \text{an element of } \mathfrak{n}^{k+1}\right)\Big|_{t=0} \\
&= d_{\mathcal{F}}P_k(X) + \text{an element of } \mathfrak{n}^{k+1}.
\end{aligned}$$

We have

$$\begin{aligned}\mathrm{Ad}(P^{-1})\Phi_*\omega_0 &= \exp\left(\mathrm{ad}\left(-\sum_{i=k}^{s} P_i\right)\right)\sum_{i=1}^{s}\omega'_{0i} \\ &= \sum_{i=1}^{s}\omega'_{0i} + \text{an element of } \mathfrak{n}^{k+1}.\end{aligned}$$

Comparing the V_k-component of (1) we get

$$\omega_{0k} + \delta_{ks}\epsilon\omega Z = \omega'_{0k} + d_{\mathcal{F}}P_k.$$

When $k = s$ the equation

$$\omega Z = \epsilon^{-1}(\omega'_{0s} - \omega_{0s}) + d_{\mathcal{F}}(\epsilon^{-1}P_s)$$

shows that ω is cohomologous to a constant cocycle.

If $k < s$, then $d_{\mathcal{F}}P_k = \phi \circ \omega_0$ for some linear map $\phi : \mathfrak{n} \to V_k$. For any $X \in \mathfrak{n}$, let $\tilde{X}$ denote the vector field on M determined by X via ρ. We have $\tilde{X}P_k = \phi(X)$ and by integrating over an integral curve γ of $\tilde{X}$ we get $P_k(\gamma(T)) - P_k(\gamma(0)) = \phi(X)T$ for all $T > 0$. Since M is compact, $\phi(X) = 0$. Therefore $d_{\mathcal{F}}P_k = 0$, so that P_k is constant on each leaf of $\mathcal{F}$. Thus P_k is constant on M by our assumption. Put $g := \exp(-P_k)$ and $Q := gP = \exp\left(\sum_{i=k+1}^{s} P_i + \text{an element of } \mathfrak{n}^{k+1}\right)$. Then

$$\begin{aligned}\omega_0 + \epsilon\omega Z &= \mathrm{Ad}(Q^{-1}g)\Phi_*\omega_0 + (L_{g^{-1}} \circ Q)^*\theta \\ &= \mathrm{Ad}(Q^{-1})\Psi_*\omega_0 + Q^*\theta\end{aligned}$$

where $\Psi_* := \mathrm{Ad}(g)\Phi_*$. □

Applying Lemma 3 repeatedly, we see that ω is cohomologous to a constant cocycle.

Finally we assume $H^1(\mathcal{F}) = H^1(\mathfrak{n})$ and prove that ρ is parameter rigid and $H^0(\mathcal{F}) = H^0(\mathfrak{n})$. Parameter rigidity of ρ follows from Proposition 1. Let $f \in H^0(\mathcal{F})$. Fix a nonzero $\phi \in H^1(\mathfrak{n})$ and denote the corresponding leafwise 1-form on M by $\tilde{\phi}$. Then $f\tilde{\phi} \in H^1(\mathcal{F}) = H^1(\mathfrak{n})$. Thus there exist $\psi \in H^1(\mathfrak{n})$ and a C^∞-function $g : M \to \mathbb{R}$ such that $f\tilde{\phi} - \tilde{\psi} = d_{\mathcal{F}}g$ where $\tilde{\psi}$ is the leafwise 1-form corresponding to ψ. Choose $X \in \mathfrak{n}$ satisfying $\phi(X) \neq 0$. Let $x(t)$ be an integral curve of $\tilde{X}$ where $\tilde{X}$ is the vector field corresponding to X. We have $f(x(t))\phi(X) - \psi(X) = \tilde{X}_{x(t)}g = \frac{d}{dt}g(x(t))$. By integrating over $[0, T]$, we get $T(f(x(0))\phi(X) - \psi(X)) = g(x(T)) - g(x(0))$. Since g is bounded, $f(x(0))\phi(X) - \psi(X)$ must be zero. Then $f(x(0)) = \frac{\psi(X)}{\phi(X)}$ and f is constant on M.

This completes the proof of Theorem 1.

4 A construction of parameter rigid actions

Let us now construct real valued cocycle rigid actions of nilpotent groups. For the structure theory of nilpotent Lie groups, see [2]. A basis $X_1, \ldots, X_n$ of $\mathfrak{n}$ is

called a *strong Malcev basis* if $\mathrm{span}_{\mathbb{R}}\{X_1,\dots,X_i\}$ is an ideal of $\mathfrak{n}$ for each i. If Γ is a lattice in N, there exists a strong Malcev basis $X_1,\dots,X_n$ of $\mathfrak{n}$ such that $\Gamma = e^{\mathbb{Z}X_1}\cdots e^{\mathbb{Z}X_n}$. Such a basis is called a *strong Malcev basis strongly based on* Γ. Let Γ and Λ be lattices in N.

Definition 1. Λ is *Diophantine with respect to* Γ if there exists a strong Malcev basis $X_1,\dots,X_n$ of $\mathfrak{n}$ strongly based on Γ and a strong Malcev basis $Y_1,\dots,Y_n$ of $\mathfrak{n}$ strongly based on Λ such that $Y_i = \sum_{j=1}^{i} a_{ij}X_j$ for every $1 \le i \le n$, where a_{ii} is Diophantine.

Let ρ be the action of Λ on $\Gamma\backslash N$ by right multiplication. First we will prove the following.

Theorem 4. *If Λ is Diophantine with respect to Γ, then every real valued C^∞ cocycle $c : \Gamma\backslash N \times \Lambda \to \mathbb{R}$ over ρ is cohomologous to a constant cocycle.*

Note that X_1 is in the center of $\mathfrak{n}$. Let $\pi : N \to \bar{N} := e^{\mathbb{R}X_1}\backslash N$ be the projection. Since $\Gamma \cap e^{\mathbb{R}X_1} = e^{\mathbb{Z}X_1}$ is a cocompact lattice in $e^{\mathbb{R}X_1}$, $\bar{\Gamma} := \pi(\Gamma) = e^{\mathbb{R}X_1}\backslash\Gamma e^{\mathbb{R}X_1}$ is a cocompact lattice in $\bar{N}$. Let $\bar{\mathfrak{n}} = \mathbb{R}X_1\backslash\mathfrak{n}$, then $\bar{X}_2,\dots,\bar{X}_n$ is a strong Malcev basis of $\bar{\mathfrak{n}}$ strongly based on $\bar{\Gamma}$.

We will see that the naturally induced map $\bar{\pi} : \Gamma\backslash N \to \bar{\Gamma}\backslash\bar{N}$ is a principal S^1-bundle. Indeed,

$$\Gamma\backslash\Gamma e^{\mathbb{R}X_1} \hookrightarrow \Gamma\backslash N \twoheadrightarrow \Gamma e^{\mathbb{R}X_1}\backslash N$$

is a principal $\Gamma\backslash\Gamma e^{\mathbb{R}X_1}$-bundle and we have

$$\Gamma\backslash\Gamma e^{\mathbb{R}X_1} \simeq \Gamma \cap e^{\mathbb{R}X_1}\backslash e^{\mathbb{R}X_1} = e^{\mathbb{Z}X_1}\backslash e^{\mathbb{R}X_1} \simeq \mathbb{Z}\backslash\mathbb{R}$$

and

$$\begin{tikzcd} e^{\mathbb{R}X_1}\backslash\Gamma e^{\mathbb{R}X_1} \arrow[r, hook] & e^{\mathbb{R}X_1}\backslash N \arrow[r, two heads] \arrow[d, two heads] & \Gamma e^{\mathbb{R}X_1}\backslash N . \\ & \bar{\Gamma}\backslash\bar{N} \arrow[ur, "\sim"] & \end{tikzcd}$$

Since $\Lambda \cap e^{\mathbb{R}X_1} = \Lambda \cap e^{\mathbb{R}Y_1} = e^{\mathbb{Z}Y_1}$ is a cocompact lattice in $e^{\mathbb{R}X_1}$, $\bar{\Lambda} := \pi(\Lambda)$ is a cocompact lattice in $\bar{N}$. $\bar{Y}_2,\dots,\bar{Y}_n$ is a strong Malcev basis of $\bar{\mathfrak{n}}$ strongly based on $\bar{\Lambda}$ and $\bar{Y}_i = \sum_{j=2}^{i} a_{ij}\bar{X}_j$ where a_{ii} is Diophantine. Therefore $\bar{\Lambda}$ is Diophantine with respect to $\bar{\Gamma}$.

Since $\bar{\pi}$ is Λ-equivariant, the action ρ of Λ when restricted to $e^{\mathbb{Z}Y_1}$, preserves fibers of $\bar{\pi}$.

Let $z \in \bar{\Gamma}\backslash\bar{N}$. Choose a point Γx in $\bar{\pi}^{-1}(z)$. Then we have a trivialization

$$\iota_{\Gamma x} : \mathbb{Z}\backslash\mathbb{R} \simeq \bar{\pi}^{-1}(z)$$

of $\bar{\pi}^{-1}(z)$ given by $\iota_{\Gamma x}(s) = \Gamma e^{sX_1}x$. Note that if we take another point $\Gamma y \in \bar{\pi}^{-1}(z)$, $\iota_{\Gamma y}^{-1} \circ \iota_{\Gamma x} : \mathbb{Z}\backslash\mathbb{R} \to \mathbb{Z}\backslash\mathbb{R}$ is a rotation.

Let $Y_1 = aX_1$ where a is Diophantine. If we identify $\bar{\pi}^{-1}(z)$ with $\mathbb{Z}\backslash\mathbb{R}$ by $\iota_{\Gamma x}$, then the action of e^{Y_1} on $\mathbb{Z}\backslash\mathbb{R}$ is $s \mapsto s + a$.

Let μ_z be the normalized Haar measure naturally defined on $\bar{\pi}^{-1}(z)$, μ the N-invariant probability measure on $\Gamma\backslash N$ and ν the $\bar{N}$-invariant probability measure on $\bar{\Gamma}\backslash\bar{N}$. For any $f \in C(\Gamma\backslash N)$,

$$\int_{\Gamma\backslash N} f d\mu = \int_{\bar{\Gamma}\backslash\bar{N}} \int_{\bar{\pi}^{-1}(z)} f d\mu_z d\nu. \tag{2}$$

Lemma 4. *ρ is ergodic with respect to μ.*

Proof. We use induction on n. If $n = 1$, ρ is an irrational rotation on $\mathbb{Z}\backslash\mathbb{R}$, hence the result is well known. In general, Let $f : \Gamma\backslash N \to \mathbb{C}$ be a Λ-invariant L^2-function with $\int_{\Gamma\backslash N} f d\mu = 0$. Since the action of $e^{\mathbb{Z}Y_1}$ on $\bar{\pi}^{-1}(z)$ is ergodic, $f|_{\bar{\pi}^{-1}(z)}$ is constant μ_z-almost everywhere. We denote this constant by $g(z)$. Then $g : \bar{\Gamma}\backslash\bar{N} \to \mathbb{C}$ is $\bar{\Lambda}$-invariant measurable function. By induction, g is constant ν-almost everywhere. By (2), this constant must be zero. Therefore f is zero μ-almost everywhere. $\square$

Let $c : \Gamma\backslash N \times \Lambda \to \mathbb{R}$ be a C^∞-cocycle over ρ. We must show that c is cohomologous to a constant cocycle $c_0 : \Lambda \to \mathbb{R}$ where $c_0(\lambda) := \int_{\Gamma\backslash N} c(x, \lambda) d\mu(x)$. Therefore we may assume that $\int_{\Gamma\backslash N} c(x, \lambda) d\mu(x) = 0$ for all $\lambda \in \Lambda$, and we will show that c is a coboundary. We prove this by induction on n. When $n = 1$, ρ is a Diophantine rotation on $\mathbb{Z}\backslash\mathbb{R}$, hence the result is well known.

Lemma 5. *For all $m \in \mathbb{Z}$,*

$$\int_{\bar{\pi}^{-1}(z)} c(s, e^{mY_1}) d\mu_z(s) = 0.$$

Proof. Fix m and put $g(z) = \int_{\bar{\pi}^{-1}(z)} c(s, e^{mY_1}) d\mu_z(s)$. For any $\lambda \in \Lambda$, cocycle equation gives $c(x, \lambda) + c(x\lambda, e^{mY_1}) = c(x, e^{mY_1}) + c(xe^{mY_1}, \lambda)$. By integrating this equation on $\bar{\pi}^{-1}(z)$, we get $g(z\pi(\lambda)) = g(z)$. Since the action of $\bar{\Lambda}$ on $\bar{\Gamma}\backslash\bar{N}$ is ergodic, g is constant. By (2), g must be zero. $\square$

Let $f : \mathbb{Z}\backslash\mathbb{R} \xrightarrow{\iota_{\Gamma x}} \bar{\pi}^{-1}(z) \xrightarrow{c(\cdot, e^{Y_1})} \mathbb{R}$. We define

$$h_z(\iota_{\Gamma x}(s)) = \sum_{k \in \mathbb{Z}\backslash\{0\}} \frac{\hat{f}(k)}{-1 + e^{2\pi i k a}} e^{2\pi i k s}.$$

Then $h_z : \bar{\pi}^{-1}(z) \to \mathbb{R}$ is C^∞, since f is C^∞ and a is Diophantine. By Lemma 5, we have

$$c(\iota_{\Gamma x}(s), e^{Y_1}) = -h_z(\iota_{\Gamma x}(s)) + h_z(\iota_{\Gamma x} e^{Y_1}).$$

If we choose another point $\Gamma e^{s_0 X_1} x \in \bar{\pi}^{-1}(z)$ to define h_z,

$$
\begin{aligned}
h_z(\iota_{\Gamma x}(s)) &= h_z(\Gamma e^{sX_1} x) = h_z(\iota_{\Gamma e^{s_0 X_1} x}(s - s_0)) \\
&= \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{-1 + e^{2\pi i k a}} \int_0^1 c(\Gamma e^{(u+s_0)X_1} x, e^{Y_1}) e^{-2\pi i k u} du \; e^{2\pi i k (s - s_0)} \\
&= \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{-1 + e^{2\pi i k a}} \int_0^1 f(u + s_0) e^{-2\pi i k u} du \; e^{2\pi i k (s - s_0)} \\
&= \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\hat{f}(k)}{-1 + e^{2\pi i k a}} e^{2\pi i k s},
\end{aligned}
$$

so that h_z is determined only by z. Define $h : \Gamma \backslash N \to \mathbb{R}$ by $h|_{\bar{\pi}^{-1}(z)} = h_z$. Then for all $x \in \Gamma \backslash N$ and $m \in \mathbb{Z}$, $c(x, e^{mY_1}) = -h(x) + h(x e^{mY_1})$.

Let $U \subset \bar{\Gamma} \backslash \bar{N}$ be open and $\sigma : U \to \bar{\pi}^{-1}(U)$ a section of $\bar{\pi}$. Then we have a trivialization $\mathbb{Z} \backslash \mathbb{R} \times U \simeq \bar{\pi}^{-1}(U)$ which sends (s, z) to $\iota_{\sigma(z)}(s) = \Gamma e^{sX_1} \sigma(z)$. Hence

$$
h(\iota_{\sigma(z)}(s)) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{-1 + e^{2\pi i k a}} \int_0^1 c(\iota_{\sigma(z)}(u), e^{Y_1}) e^{-2\pi i k u} du \; e^{2\pi i k s}
$$

on $\bar{\pi}^{-1}(U)$. The following lemma shows h is C^∞ on $\Gamma \backslash N$.

Lemma 6. *Let $U \subset \mathbb{R}^n$ be open and let $f : \mathbb{Z} \backslash \mathbb{R} \times U \to \mathbb{R}$ be a C^∞-function. Define*

$$
h(s, z) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{-1 + e^{2\pi i k a}} \widehat{f_z}(k) e^{2\pi i k s}
$$

where $f_z(u) = f(u, z)$. Then $h : \mathbb{Z} \backslash \mathbb{R} \times U \to \mathbb{R}$ is C^∞.

Proof. Let V be open such that $\bar{V} \subset U$ and $\bar{V}$ is compact. We will show that h is C^∞ on $\mathbb{Z} \backslash \mathbb{R} \times V$. Choose constants $C, \alpha > 0$ such that $|-1 + e^{2\pi i k a}| \geq C|k|^{-\alpha}$ for all $k \in \mathbb{Z} \setminus \{0\}$.

We will first prove that h is continuous. Since for any $m \in \mathbb{Z}_{>0}$,

$$
\frac{\partial^m f_z}{\partial s^m}(s) = \sum_{k \in \mathbb{Z}} (2\pi i k)^m \widehat{f_z}(k) e^{2\pi i k s}
$$

in $L^2(\mathbb{Z} \backslash \mathbb{R})$,

$$
\begin{aligned}
\left\| \frac{\partial^m f_z}{\partial s^m} \right\|_2^2 &= \sum_{k \in \mathbb{Z}} |(2\pi i k)^m \widehat{f_z}(k)|^2 \\
&\geq (2\pi)^{2m} |k|^{2m} |\widehat{f_z}(k)|^2 \geq |k|^{2m} |\widehat{f_z}(k)|^2.
\end{aligned}
$$

Since $\left\| \frac{\partial^m f_z}{\partial s^m} \right\|_2 = \left(\int_0^1 \left| \frac{\partial^m}{\partial s^m} f(s, z) \right|^2 ds \right)^{\frac{1}{2}}$ is continuous in z, there exists $M > 0$ such that $\left\| \frac{\partial^m f_z}{\partial s^m} \right\|_2 < M$ for every $z \in \bar{V}$. Hence for all $k \in \mathbb{Z}$ and $z \in \bar{V}$,

$|k|^m|\widehat{f}_z(k)| \le M$. Therefore, for any $z \in \bar{V}$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}\setminus\{0\}} \Big|\frac{1}{-1+e^{2\pi ika}}\widehat{f}_z(k)e^{2\pi iks}\Big| &\le C^{-1}\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{1}{|k|^2}|k|^{\alpha+2}|\widehat{f}_z(k)| \\ &\le C^{-1}M\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{1}{|k|^2} < \infty.\end{aligned}$$

This implies continuity of h on $\mathbb{Z}\backslash\mathbb{R}\times\bar{V}$.

We have
$$\frac{\partial h}{\partial s}(s,z) = \sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{2\pi ik}{-1+e^{2\pi ika}}\widehat{f}_z(k)e^{2\pi iks}.$$

Thus a similar argument shows that $\frac{\partial h}{\partial s}$ is continuous.

Let $z=(z_1,\dots,z_n)$. For any $z\in\bar{V}$,

$$\begin{aligned}\Big|\frac{\partial}{\partial z_j}\left(\frac{1}{-1+e^{2\pi ika}}\widehat{f}_z(k)e^{2\pi iks}\right)\Big| &= \Big|\frac{1}{-1+e^{2\pi ika}}\widehat{\frac{\partial f}{\partial z_j}(\cdot,z)}(k)e^{2\pi iks}\Big| \\ &\le C^{-1}\frac{1}{|k|^2}|k|^{\alpha+2}\Big|\widehat{\frac{\partial f}{\partial z_j}(\cdot,z)}(k)\Big| \\ &\le C^{-1}M'\frac{1}{|k|^2} \ \in L^1(\mathbb{Z}\setminus\{0\}).\end{aligned}$$

Thus
$$\frac{\partial h}{\partial z_j}(s,z) = \sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{1}{-1+e^{2\pi ika}}\widehat{\frac{\partial f}{\partial z_j}(\cdot,z)}(k)e^{2\pi iks}.$$

Hence $\frac{\partial h}{\partial z_j}$ is continuous by an argument similar to those above. For higher derivatives of h, continue this procedure. $\square$

Set $c_1(x,\lambda)=c(x,\lambda)+h(x)-h(x\lambda)$. $c_1:\Gamma\backslash N\times\Lambda\to\mathbb{R}$ is a C^∞-cocycle and $c_1(x,e^{mY_1})=0$. Thus for any $\lambda\in\Lambda$, cocycle equation implies $c_1(x,\lambda)=c_1(xe^{Y_1},\lambda)$. Since the action of $e^{\mathbb{Z}Y_1}$ on $\bar{\pi}^{-1}(z)$ is ergodic, $c_1(x,\lambda)$ is constant on $\bar{\pi}^{-1}(z)$. Therefore we can define a cocycle $\bar{c}:\bar{\Gamma}\backslash\bar{N}\times\bar{\Lambda}\to\mathbb{R}$ by $\bar{c}(\bar{\pi}(x),\pi(\lambda))=c_1(x,\lambda)$. Indeed, if $\bar{\pi}(x)=\bar{\pi}(y)$ and $\pi(\lambda)=\pi(\lambda')$, then there exists a $m\in\mathbb{Z}$ with $\lambda=e^{mY_1}\lambda'$, so that

$$c_1(x,\lambda)=c_1(x,e^{mY_1}\lambda')=c_1(xe^{mY_1},\lambda')=c_1(y,\lambda').$$

Furthermore,

$$\begin{aligned}\int_{\bar{\Gamma}\backslash\bar{N}}\bar{c}(x,\pi(\lambda))d\nu(z) &= \int_{\bar{\Gamma}\backslash\bar{N}}\int_{\bar{\pi}^{-1}(z)}c_1(s,\lambda)d\mu_z(s)d\nu(z) \\ &= \int_{\Gamma\backslash N}c_1(x,\lambda)d\mu(x)=0.\end{aligned}$$

By induction, there exists a C^∞-function $P : \bar{\Gamma}\backslash\bar{N} \to \mathbb{R}$ such that $\bar{c}(z, \pi(\lambda)) = -P(z) + P(z\pi(\lambda))$. Put $Q = P \circ \bar{\pi}$. Then $c_1(x, \lambda) = \bar{c}(\bar{\pi}(x), \pi(\lambda)) = -Q(x) + Q(x\lambda)$. This proves Theorem 4.

Let $\tilde{\rho} : M \times N \to M$ be the suspension of $\rho : \Gamma\backslash N \times \Lambda \to \Gamma\backslash N$ where $M = \Gamma\backslash N \times_\Lambda N$ is a compact manifold. Then $\tilde{\rho}$ is locally free and let $\mathcal{F}$ be its orbit foliation. We have

$$H^1(\mathcal{F}) \simeq H^1(\Lambda, C^\infty(\Gamma\backslash N))$$

by [6] where the right hand side is the first cohomology of Λ-module $C^\infty(\Gamma\backslash N)$ obtained by ρ. It is easy to prove that $\mathrm{Hom}(\Lambda, \mathbb{R}) \to H^1(\Lambda, C^\infty(\Gamma\backslash N))$ is injective. By Theorem 4,

$$H^1(\Lambda, C^\infty(\Gamma\backslash N)) = \mathrm{Hom}(\Lambda, \mathbb{R}).$$

Lemma 7. $\dim \mathrm{Hom}(\Lambda, \mathbb{R}) = \dim H^1(\mathfrak{n})$.

Proof. Recall that $[N, N]\backslash\Lambda[N, N]$ is a cocompact lattice in $[N, N]\backslash N$ and that $[\Lambda, \Lambda]\backslash (\Lambda \cap [N, N])$ is finite. Since

$$0 \to [\Lambda, \Lambda]\backslash (\Lambda \cap [N, N]) \to [\Lambda, \Lambda]\backslash\Lambda \to [N, N]\backslash\Lambda[N, N] \to 0$$

is exact, we have

$$\mathrm{rank}[\Lambda, \Lambda]\backslash\Lambda = \mathrm{rank}[N, N]\backslash\Lambda[N, N] = \dim[N, N]\backslash N.$$

Thus

$$\begin{aligned} \dim \mathrm{Hom}(\Lambda, \mathbb{R}) &= \dim \mathrm{Hom}([\Lambda, \Lambda]\backslash\Lambda, \mathbb{R}) \\ &= \mathrm{rank}[\Lambda, \Lambda]\backslash\Lambda \\ &= \dim[N, N]\backslash N \\ &= \dim \mathrm{Hom}_{\mathbb{R}}([\mathfrak{n}, \mathfrak{n}]\backslash\mathfrak{n}, \mathbb{R}) \\ &= \dim H^1(\mathfrak{n}). \end{aligned}$$

□

Therefore we obtain

$$H^1(\mathcal{F}) = H^1(\mathfrak{n}).$$

This proves Theorem 3.

5 Existence of Diophantine lattices

Let $\mathfrak{n}_{\mathbb{Q}}$ be a rational structure of $\mathfrak{n}$. We construct Diophantine lattices when $\mathfrak{n}_{\mathbb{Q}}$ admits a graduation. Namely, we assume that $\mathfrak{n}_{\mathbb{Q}}$ has a sequence V_i of $\mathbb{Q}$-subspaces such that $\mathfrak{n}_{\mathbb{Q}} = \bigoplus_{i=1}^k V_i$ and $[V_i, V_j] \subset V_{i+j}$. Let $X_1, \ldots, X_n$ be a $\mathbb{Q}$-basis of $\mathfrak{n}_{\mathbb{Q}}$ such that $X_1, \ldots, X_{i_1} \in V_k$, $X_{i_1+1}, \ldots, X_{i_2} \in V_{k-1}, \ldots,$

$X_{i_{k-1}+1}, \ldots, X_n \in V_1$. Then $X_1, \ldots, X_n$ is a strong Malcev basis of $\mathfrak{n}$ with rational structure constants. Multiplying $X_1, \ldots, X_n$ by a integer if necessary, we may assume that $\Gamma := e^{\mathbb{Z}X_1} \cdots e^{\mathbb{Z}X_n}$ is a cocompact lattice in N. Let α be a root of a irreducible polynomial of degree $k+1$ over $\mathbb{Q}$. Since $\alpha, \alpha^2, \ldots, \alpha^k$ are irrational algebraic numbers, they are Diophantine. If we define a linear map $\varphi : \mathfrak{n} \to \mathfrak{n}$ by $\varphi(X) = \alpha^i X$ for $X \in V_i \otimes \mathbb{R}$, then φ is an automorphism of Lie algebra $\mathfrak{n}$. Put $Y_i = \varphi(X_i)$. $Y_1, \ldots, Y_n$ is a strong Malcev basis of $\mathfrak{n}$ strongly based on $\Lambda := e^{\mathbb{Z}Y_1} \cdots e^{\mathbb{Z}Y_n}$. Thus Λ is Diophantine with respect to Γ.

Acknowledgement

The author would like to thank his advisor, Masayuki Asaoka, for helpful comments.

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